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First-Quantized Para-Particle Theory

Steven French[1]

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The general formalism of first-quantized para-particle theory is presented in a way that clearly exposes the more important group-theoretic foundations. This is then used to give a simple and straightforward solution to a problem concerning possible changes in a particle's statistics through collisions, first posed by Pauli in 1927, and to show that para-particle statistics are different from those obeyed by particles in a "para-gas." These arguments accord with the overall aim of demonstrating that para-particle theory is not as theoretically impenetrable as is usually thought, but in fact possesses a kind of generalized elegance.

1. INTRODUCTION

I present here a simple solution to a problem first posed by Pauli in 1927. This is the question of whether it is possible for a particle to change its statistical type through collisions with an N-particle system. Any answer to this question will obviously involve a consideration of which transitions are allowed between the particle's states and will therefore have a fundamental bearing on the problem of the concrete statistical behavior of para-particles. In the final section of this work I demonstrate that this behavior cannot, in fact, be captured by the well-known "para-gas" theory, in which the average number of particles in a group of states is dependent upon a parameter giving the maximum number of particles that can occupy any given state.

2. THE GENERAL FORMALISM

Consider, for simplicity, a system of N indistinguishable particles of the same kind and with zero spin. A particular ket $|f\rangle$ for such an assembly

[1]Centro de Logica, Universidade Estadual de Campinas, Caixa Postal 6133, 13100, Campinas, S.P., Brazil.

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can be obtained in the well-known way (Dirac, 1978, p. 207) by taking the tensor product of kets for each particle by itself:

$$|f\rangle = |a_1^1\rangle \otimes |a_2^2\rangle \cdots \otimes \cdots |a_N^N\rangle$$
$$\equiv |a_1^1 \cdots a_N^N\rangle \tag{1}$$

where the subscripts label the particles and the superscripts the states.

These product functions and their suitably permuted variants span a Hilbert space $\mathscr{H}_N$ constructed from N single product spaces:

$$\mathscr{H}_N = \mathscr{H}_1 \otimes \mathscr{H} \otimes \cdots \mathscr{H}_1 \quad (N \text{ times})$$

Inside each $\mathscr{H}_N$ one can define unitary operators describing the permutation of particles of the same kind. These particle permutation operators P_i form a group, usually denoted by S_N, and can be defined by

$$P_i = \begin{pmatrix} i \\ r_i \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \cdots \\ r_1 & r_2 & r_3 & \cdots \end{pmatrix}$$

The application of such an operator to a ket for the assembly produces another ket of the form (1) spanning $\mathscr{H}_N$ (Dirac, 1978, p. 208).

We now introduce the following postulate.

The Indistinguishability Postulate (IP). If a particle permutation P_i is applied to any ket for the assembly, then there is no way of distinguishing the resulting permuted ket from the original unpermuted one by means of any observation at any time (Greenberg and Messiah, 1964, p. 250; Hartle and Taylor, 1969, p. 2044).

More formally, this requirement can be expressed in terms of expectation values:

$$\langle P_i f | Q | P_i f \rangle = \langle f | Q | f \rangle$$

where the state $P_i|f\rangle$ is abbreviated to $|P_i f\rangle$.

Since P_i is a unitary operator, we have

$$\langle f | Q | f \rangle = \langle f | P_i^{-1} Q P_i | f \rangle \tag{2}$$

where Q is some suitably defined Hermitian operator.

Since this must hold for any vector $|f\rangle$, it will hold with $|f\rangle$ replaced by any linear combination of vectors. Applying this to the two superpositions $|f\rangle + \alpha|g\rangle$ and $|f\rangle + i\alpha|g\rangle$ gives

$$\langle f | Q | g \rangle = \langle f | P_i^{-1} Q P_i | g \rangle, \qquad f, g \in \mathscr{H}_N$$

or

$$[P_i, Q] = 0$$

That is, all physical observables must be particle permutation invariant.

In addition to particle permutation operators, one can also define *place* permutation operators $\bar{P}_i$, which permute the state labels and can be defined by

$$\bar{P}_i = \begin{pmatrix} i \\ r_i \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \cdots & N \\ r_1 & r_2 & r_3 & \cdots & r_N \end{pmatrix}$$

[See Ref. Dirac (1978, p. 217) for a discussion of the formal differences between the P_i and the $\bar{P}_i$.]

As is well known, these two kinds of permutation operator are related by

$$\bar{P}_i|f\rangle = P_i^{-1}|f\rangle$$

and whereas the P_i cannot be regarded as observables, self-adjoint functions of the $\bar{P}_i$ *can* (Dirac, 1978, Landshoff and Stapp, 1967, p. 73; Stolt and Taylor, 1970a, p. 6).

Having identified the observables for N indistinguishable particles as being that subset of observables for N labeled particles that commute with the P_i, this following from IP, the next step is obviously to establish the correspondence between the physical states of the system and the vectors of $\mathscr{H}_N$. Since any state can be completely characterized by specifying the expectation values of all observables, these vectors must possess the following simple properties:

1. Two vectors $|f\rangle$ and $|f'\rangle$ representing the same state must give the same expectation value for all observables, i.e.,

$$\langle f|Q|f\rangle = \langle f'|Q|f\rangle, \qquad \text{all } Q \text{ with } [Q, P_i] = 0$$

Thus, IP implies that whenever a vector $|f\rangle$ in $\mathscr{H}_N$ corresponds to some physical state, then the vector $P_i|f\rangle$ for any i must correspond to the same state. It is at this point, of course, that the "ordinary" and para-particle theories diverge, since the fundamental assumption of the former that every physically distinct state must correspond to some unique ray in $\mathscr{H}_N$ is dropped in the latter and the possibility is allowed of a single state of the assembly of particles corresponding to some larger collection of vectors in this space (Hartle and Taylor, 1969, p. 2045; Stolt and Taylor, 1970a, p. 1).

2. Two vectors $|f''\rangle$ and $|f'''\rangle$ representing different states must give different expectation values for some observable, i.e.,

$$\langle f'''|Q|f'''\rangle \neq \langle f''|Q|f''\rangle, \qquad \text{some } Q \text{ with } [Q, P_i] = 0$$

The apparatus of group theory can then be used to determine the sets of vectors possessing these qualities (Hartle and Taylor, 1969).

We begin by considering an arbitrary vector $|f\rangle$ in $\mathscr{H}_N$ and the $N!$-dimensional subspace spanned by the vectors $P_i|f\rangle$ for all permutations i

in the group S_N. This subspace can be decomposed into irreducible subspaces invariant under the P_i. Each subspace carries an irreducible representation D^{μ} of S_N and the number of subspaces carrying any irreducible representation D^{μ} is equal to the dimension N^{μ} of D^{μ}. The problem now, of course, is to construct basis functions for the various irreducible representations of S_N.

The procedure for doing this is presented in Hammermesh (1962, pp. 85 and 111), and we give here a streamlined version suitable for our purposes.

Note, first, that in general any function $|f\rangle$ is expressible as the sum of functions that can act as basis functions in the various irreducible representations:

$$|f\rangle = \sum_{\nu} \sum_{i=1}^{n} |f_i\rangle^{(\nu)} \tag{3}$$

The basis functions for the νth irreducible (unitary) representation satisfy

$$Q_R |f_i\rangle^{(\nu)} = \sum_{j}^{(\nu)} D_{ji}^{(\nu)}(R) |f_j\rangle^{(\nu)}$$

where Q_R is any operator of the group considered.

The necessary and sufficient condition a given function must satisfy in order that it may belong to the ith row of a given representation is

$$\sum_{R} D_{ii}^{(\nu)*}(R) Q_R |f_i\rangle^{(\nu)} = \frac{h}{n_\nu} |f_i\rangle^{(\nu)} \tag{4}$$

Thus, given a function $|f_i\rangle^{(\nu)}$ that satisfies the above requirements, one can associate with it $n_\nu - 1$ "partners" given by

$$|f_l\rangle^{(\nu)} = \frac{n_\nu}{h} \sum_{R} D_{li}^{(\nu)*}(R) Q_R |f_i\rangle^{(\nu)}$$

so that the set of functions satisfies (4).

Returning to (3), the question now is, given the function $|f\rangle$, how does one find the $|f_i\rangle^{(\nu)}$ in the first place? In other words, how does one resolve the given function into a sum of functions, each of which belongs to a particular row of some irreducible representation?

Applying the projection operators

$$\Pi_{jj}^{(\mu)} = \frac{n_\mu}{h} \sum_{R} D_{jj}^{(\mu)*}(R) Q_R$$

to equation (3) gives the desired functions:

$$|f_j\rangle^{(\mu)} = \frac{n_\mu}{h} \sum_{R} D_{jj}^{(\mu)*}(R) Q_R |f\rangle$$

Then, applying the transfer operator

$$\Pi_{ij}^{(\nu)} = \frac{n_\nu}{h} \sum_R D_{ij}^{(\nu)*}(R) Q_R$$

gives the $n_\nu - 1$ partners such that the whole set satisfies equation (4), thus giving the functions $|f_i\rangle^{(\nu)}$ that form a basis for the νth irreducible representation.

However, it can be shown (French, 1985, p. 149) that

$$\Pi_{ij}\Pi_{jj} = \Pi_{ij}$$

and thus the first stage above can be dispensed with, a result Hartle and Taylor (1969) clearly knew, but failed to make explicit. Applying the Π_{ij} to the arbitrary state vector $|f\rangle$ with the P_i in the role of the Q_R, one then obtains the basis functions $|f_i\rangle$:

$$|f_i\rangle = \Pi_{ij}|f\rangle$$

$$= \frac{n_\mu}{h} \sum_R D_{ij}^{(\mu)*}(R) P(R)|f\rangle$$

These functions $\Pi_{ij}|f\rangle$ support representations of both the P's and the $\bar{P}$'s and therefore satisfy the equations

$$P_i|f_i\rangle = \sum_i |f_j\rangle D_{ji}^{(\mu)}(R) \tag{5}$$

$$\bar{P}_i|f_i\rangle = \sum_i |f_j> D_{ij}^{(\mu)}(R) \tag{6}$$

Equation (5) implies that for fixed j and for $i = 1, \ldots, N$, these functions span an irreducible subspace invariant under the P_i. Within this subspace—or generalized ray, as it has been called (Greenberg and Messiah, 1964, p. 251)—the function $|f_i\rangle$ transforms as the ith basis function of the irreducible representation $D^{(\mu)}$. The fact that the P_i leave any function in the same subspace, or generalized ray, reflects the fact that two state functions that differ by a permutation of the particle labels must represent the same state, as we noted above.

Equation (6) implies, similarly, that for fixed i and $j = 1, \ldots, N$, the functions span a subspace invariant under the $\bar{P}_i$ and within this space $|f_i\rangle$ transforms as the jth basis function of the irreducible representation $D^{(\mu)}$. Clearly in general the place permutation operators carry a function from one subspace to another, which is consistent with the fact that place permutations *can* change the physical state. [For an illustration of the complementary role of the P's and $\bar{P}$'s, see Stolt and Taylor (1970a, p. 9).]

It is worth noting that in the case of the one-dimensional symmetric and antisymmetric representations, $D = 1$ or ± 1. Thus, P_i and $\bar{P}_i$ are the same for the symmetric and antisymmetric vectors and so the distinction between place and particle permutations is not manifested in the case of "ordinary" bosons and fermions.

Using IP and Schur's lemma it can easily be shown that these basis functions $|f_i\rangle$ indeed satisfy the two requirements for vectors to represent states (Hartle and Taylor, 1969, p. 2045).

The next step is to consider which irreducible representations correspond to states. In the boson and fermion cases this is perfectly straightforward, since only the rays of the totally symmetric or totally antisymmetric representations are acceptable. The case of para-particles requires a little more elaboration, however.

If $\mathscr{E}_\mu$ is defined to be the subspace spanned by the basis functions $|f_i\rangle$ (with μ fixed), then the Hilbert space $\mathscr{H}_N$ can be decomposed into a number of subspaces:

$$\mathscr{H}_N = \bigoplus_\mu \mathscr{E}_\mu$$

Although $\mathscr{E}_\mu$ is defined with respect to a definite basis, it can be seen that it is actually basis-independent. It contains all functions associated with the irreducible representation D^μ and since every pure state must be associated with a definite irreducible representation, each such state is represented by a generalized ray of vectors contained in some $\mathscr{E}_\mu$.

However, although every pure state is associated with a definite irreducible representation, not every one of the latter is associated with an attainable state, as the example of bosons and fermions makes clear. In general the statistical type of a para-particle is identified by specifying the set of all irreducible representations in each S_N that corresponds to attainable states of the particle.

It was demonstrated by Hartle and Taylor (1969) that not every family of irreducible representations corresponds to a possible statistical type. These results were then extended by Stolt and Taylor (1970b, c), who showed that all possible types can be divided into two kinds: those of finite and those of infinite order. The former can be further classified into para-bosons and para-fermions of order $p = 1, 2, 3, \ldots$, but there are infinitely many types of para-particles of infinite order.

The space appropriate to a para-particle of a given statistical type T is the subspace of $\mathscr{H}_N$ containing just those vectors that correspond to allowed irreducible representations. If the set of all such representations associated with the attainable N-particle states of a particle of type T is denoted by T_N, then the appropriate space is simply the direct sum (Stolt

and Taylor, 1970a, p. 10)

$$\mathscr{H}_N^T = \bigoplus_{\mu \in T_N} \mathscr{E}_\mu$$

The final step is to eliminate the "generalized ray" and restore the usual connection between states and rays in quantum mechanics (Hartle and Taylor, p. 2046; Stolt and Taylor, 1970a, pp. 10–11).

We note that each $\mathscr{E}_\mu$ can be further decomposed,

$$\mathscr{E}_\mu = \bigoplus_{i=1}^{n_\mu} \mathscr{E}_{\mu,i}$$

where $\mathscr{E}_{\mu,i}$ is the space spanned by the $|f_i\rangle$ for fixed μ and i. Since each "generalized ray" (corresponding to an n_μ-dimensional subspace of $\mathscr{E}_\mu$) has a unique one-dimensional intersection with each $\mathscr{E}_{\mu,i}$, one could always label states of symmetry D^μ by vectors from one specific $\mathscr{E}_{\mu,i}$, $\mathscr{E}_{\mu,1}$ say, and then each state is labeled by a unique ray. The $\mathscr{E}_{\mu,i}$ for $i \neq 1$ can be ignored because any $\mathscr{E}_{\mu,i}$, and in particular $\mathscr{E}_{\mu,1}$, is invariant under all observables. This follows from the fact that all observables commute with the P's. In particular, since this invariance applies to the Hamiltonian, all representative vectors will remain in a particular $\mathscr{E}_{\mu,i}$ for all times if they are chosen there initially.

Thus, the formalism in which states of symmetry D^μ are represented by generalized rays in $\mathscr{E}_\mu$ can be replaced by one in which the same states are represented by rays in $\mathscr{E}_{\mu,i}$. For a particle of type T the new, smaller space is

$$\mathscr{H}_{N,\text{ray}}^T = \bigoplus_{\mu \in T_N} \mathscr{E}_{\mu,1}$$

The essential point behind this elimination is that any state represented by a multidimensional subspace with basis $\{|f_i\rangle\}$ can be represented arbitrarily by the number one basis vector $|f_1\rangle$. Every state can then be labeled by a unique ray in $\mathscr{E}_{\mu,1}$ rather than a "generalized ray" in $\mathscr{E}_{\mu,i}$, thus restoring the usual connection between states and rays.

This not only allows one to regain all the usual results of "ordinary" quantum mechanics, it also permits a direct comparison between the first- and second-quantized formalisms. Stolt and Taylor have demonstrated that for every para-particle of type T of finite order, there exists a natural isomorphism between $\mathscr{H}_{N,\text{ray}}^T$ and the N-particle space for the corresponding para-field. Thus, they established that every type of second-quantised para-field is equivalent to a unique first-quantized para-particle of finite order and vice versa (Stolt and Taylor, 1970a).

Having set out the basic formalism, and, one hopes, demonstrated that it is not quite as forbidding as is sometimes thought, we are now in a

position to discuss the possibility of transitions between states of different statistical type.

3. TRANSITIONS BETWEEN STATES CORRESPONDING TO REPRESENTATIONS OF DIFFERENT STATISTICAL TYPE

We shall consider the simple case of a three-particle assembly. Thus we begin by generating the basis functions for three particle states (French, 1985, pp. 156–160).

Using the procedure given above together with the real orthogonal matrices of the irreducible representations of the permutation group S_3 and expressions for the P's and $\bar{P}$'s in terms of their effect upon a state vector of the form $|f\rangle$, we obtain, very straightforwardly, the four "mixed symmetry" basis functions spanning the so-called "triangular" subspaces [compare with those of Hartle and Taylor (1969)]:

$$f'_{11} = \frac{1}{2\sqrt{3}}(2|a_1^1 a_2^2 a_3^3\rangle + 2|a_2^1 a_1^2 a_3^3\rangle - |a_3^1 a_2^2 a_1^3\rangle - |a_1^1 a_3^2 a_2^3\rangle - |a_2^1 a_3^2 a_1^3\rangle - |a_3^1 a_1^2 a_2^3\rangle)$$

$$f'_{21} = \tfrac{1}{2}(|a_3^1 a_2^2 a_1^3\rangle - |a_1^1 a_3^2 a_2^3\rangle - |a_2^1 a_3^2 a_1^3\rangle + |a_3^1 a_1^2 a_2^3\rangle)$$

$$f''_{12} = \tfrac{1}{2}(|a_3^1 a_2^2 a_1^3\rangle - |a_1^1 a_3^2 a_2^3\rangle + |a_2^1 a_3^2 a_1^3\rangle - |a_3^1 a_1^2 a_2^3\rangle)$$

$$f''_{22} = \frac{1}{2\sqrt{3}}[2|a_1^1 a_2^2 a_3^3\rangle - 2|a_2^1 a_1^2 a_3^3\rangle + |a_3^1 a_2^2 a_1^3\rangle + |a_1^1 a_3^2 a_2^3\rangle - |a_2^1 a_3^2 a_1^3\rangle - |a_3^1 a_1^2 a_2^3\rangle)$$

The symmetric and antisymmetric functions spanning the one-dimensional symmetric and antisymmetric subspaces are simply

$$f_S = \tfrac{1}{6}(|a_1^1 a_2^2 a_3^3\rangle + |a_2^1 a_1^2 a_3^3\rangle + |a_3^1 a_2^2 a_1^3\rangle + |a_1^1 a_3^2 a_2^3\rangle + |a_2^1 a_3^2 a_1^3\rangle + |a_3^1 a_1^2 a_2^3\rangle)$$

$$f_A = \tfrac{1}{6}(|a_1^1 a_2^2 a_3^3\rangle - |a_2^1 a_1^2 a_3^3\rangle - |a_3^1 a_2^2 a_1^3\rangle - |a_1^1 a_3^2 a_2^3\rangle + |a_2^1 a_3^2 a_1^3\rangle + |a_3^1 a_1^2 a_2^3\rangle)$$

These six basis functions span the six-dimensional subspace $\mathscr{E}$ of $\mathscr{H}_3$, which can be decomposed into irreducible subspaces invariant under S_3:

$$\mathscr{E} = \mathscr{E}_S \oplus (\mathscr{E}_{I'} \oplus \mathscr{E}_{I''}) \oplus \mathscr{E}_A$$

$\mathscr{E}_S$ and $\mathscr{E}_A$ are the one-dimensional subspaces defined, respectively, by the symmetric and antisymmetric functions f_S and f_A. The remaining four-dimensional space splits into two irreducible subspaces transforming under the same two-dimensional representation of S_3 (that of the triangular Young diagram, hence the name "triangular" subspace). Our choice for these two subspaces is given by f'_{11} and f'_{21} spanning $\mathscr{E}_{t'}$ and f''_{12} and f''_{22} spanning $\mathscr{E}_{t''}$.

As $|f\rangle$ runs over $\mathscr{H}_3$ it can be decomposed in this way into three sectors:

$$\mathscr{H}_S = \mathscr{H}_S \oplus \mathscr{H}_t \oplus \mathscr{H}_A$$

The irreducible subspaces of $\mathscr{H}_S$ and $\mathscr{H}_A$ are one-dimensional, whereas those of $\mathscr{H}_t$ are two-dimensional. Pure states may be represented by any irreducible subspace in any of these sectors. Two states represented by subspaces of different representations are physically distinct, as are the states represented by different subspaces of the same representation. These points have an obvious importance for what is to follow.

Given IP as expressed in the form $[P, Q] = 0$, Schur's lemma implies that all matrix elements of the observable Q connecting different representations are zero. It then follows that if there exist states corresponding to subspaces of different representations of S_N, then they must be separated by a superselection rule (Greenberg and Messiah, 1969, p. 251). In other words, transitions between such states, for example, those corresponding to the symmetric and antisymmetric representations, or the symmetric and "triangular," are absolutely forbidden.

However, states corresponding to subspaces of the *same* irreducible representation, such as $\mathscr{E}_{t'}$ and $\mathscr{E}_{t''}$, where the same representation is repeated twice, can be connected. Transitions *can* therefore occur between such states, although with certain two-body collisions these transitions will also be forbidden, as we shall see.

The situation in the general N-particle case is quite analogous. The Hilbert space then decomposes into a number of subspaces and transitions between states corresponding to certain of these subspaces—i.e., those of different representations—are not allowed. Thus, restrictions are imposed by IP such that certain states are rendered inaccessible to particles of the same species. If we consider the time evolution of the system as effected by some Hamiltonian, then we can conclude that what IP effectively says is that if the system starts in one irreducible representation, then it will always remain in that representation.

These are important points, which we shall have cause to recall in our discussion of Pauli's problem below.

Before we proceed any further, however, we have to deal with the question, first raised by Steinmann (1966), following a result suggested by Pauli (1958, p. 110), of whether para-particle theory is consistent with the

well-known cluster principle. This states that systems sufficiently separated in space may be treated as isolated systems, or, in other words, that the presence of particles of Mars, say, should not affect the results of experiments performed on Earth.

The core of Steinmann's objection can be summarized as follows: Consider a three-particle system consisting of a two-particle cluster on Earth and the other particle on Mars. If the symmetry type of particles 1 and 2 on Earth, when 3 is on Mars, is measured, then, the argument runs, the functions f'_{11} and f'_{21} can be distinguished, yet they are supposed to be indistinguishable, since they support the same irreducible representation of S_3. Hence there is an inconsistency.

That this argument is fallacious was clearly shown by Hartle and Taylor, but we shall demonstrate this again, in a slightly different way, in order to bring out some points that will be useful later.

We begin by noting that we have chosen our basis functions above to be eigenvectors of P_2, which interchanges particles labeled 1 and 2. (It can be shown that the basis functions chosen by Hartle and Taylor are not eigenvalues of P_2, but that this does not affect the substance of their arguments.) The eigenvalues of this operator can easily be compared with those of the place permutation operator $\bar{P}_2$ (Table I). An immediate consequence of these results is that eigenvalues of $\bar{P}_2$ serve to distinguish the symmetry types for "clusters" of para-particles, whereas those of the corresponding particle permutation operator *do not.* [Hartle and Taylor believed that it was possible to have basis vectors for a "triangular" representation of S_3 that are eigenvectors of P_2 with the same eigenvalues. However, this is clearly not true. See French (1985, pp. 163–165).]

Thus, the basic fallacy in Steinmann's argument is the supposition that the symmetry type of a two-particle cluster is determined by the eigenvalue of P_2. He assumed that because this operator possesses opposite eigenvalues for f'_{11} and f'_{21}, for example, then the symmetry types for two-particle clusters are different for f'_{11} and f'_{21}. However, this is not true. It can easily be shown

Table I. Eigenvalues

Basis functions	Eigenvalues of P_2	Eigenvalues of $\bar{P}_2$
f_S	+1	+1
f_A	−1	−1
f'_{11}	+1	+1
f'_{21}	−1	+1
f''_{12}	+1	−1
f''_{22}	−1	−1

that both f'_{11} and f'_{21} give *symmetric* type functions for two-particle clusters and that f''_{12} and f''_{22} both give *antisymmetric* types, although they also possess opposite eigenvalues for P_2 (French, 1985, pp. 166–168). Thus, measurements of the symmetry type of two-particle clusters do not distinguish between f'_{11} and f'_{21}, which must be indistinguishable, since they belong to the same irreducible subspace, but between f'_{11} and f''_{22}, which is perfectly consistent, since these functions belong to different irreducible subspaces (Hartle and Taylor, 1969, p. 2048).

We are now finally in a position to address Pauli's 1927 question.

4. PAULI'S PROBLEM

Pauli (1927) considered the collision between an electron and some N-particle system and concluded that such a collision could induce transitions between states of the system corresponding to different irreducible representations of S_N. Thus, for example, a system of bosons could be transformed into a system of fermions via some mixed symmetry state.

We have already noted that transitions between states corresponding to subspaces of different representations of S_N are not allowed, since a superselection rule operators between such states. This means that transitions between the particles of Pauli's N-particle system cannot occur between states corresponding to the symmetric and antisymmetric representations, for example (Wigner, 1927, p. 883; Heisenberg, 1930, p. 156), nor between the symmetric and triangular representations. No such rule exists for states corresponding to equivalent representations, and transitions between such states *are* possible, in general.

We shall now demonstrate, first, that these latter transitions cannot in fact occur for two-body collisions of a certain kind, but that, second, they may take place for collisions involving three particles.

We assume that all states a_1^1, $a_1^{1'}$, a_2^2, $a_2^{2'}$, a_3^3, $a_3^{3'}$, are orthogonal and that a_1^1, $a_1^{1'}$, a_2^2, $a_2^{2'}$ are wave functions localized on the Earth, whereas a_3^3, $a_3^{3'}$ are localized on Mars. Thus a_1 and a_2 represent the Earth coordinates and a_3 the Mars coordinate.

According to the cluster principle, terms of the form $a_1^1 a_1^3$ or $a_1^2 a_1^3$ are then identically zero—there is no overlap between the wave functions localized on the Earth and on Mars, respectively.

The first step in our demonstration is to show that any symmetric function of the observables Q can be expanded in terms of the $\bar{P}$'s. This follows directly from Theorem 3.4A of Weyl (1946, Chapter IV). Dirac (1978, p. 218) sketched a proof for the particular case of the observable being some perturbing energy V defined by $\langle f|VP|f\rangle$. However a more general proof can be given as follows.

Theorem. The following relation holds:

$$\langle f'|P_jQP_i|f\rangle = \sum_i \langle f'|QP_i|f\rangle\langle f|P_j\bar{P}_iP_l|f\rangle$$

where $|f'\rangle$ and $|f\rangle$ are kets for the assembly of the form given at the beginning of Section 2.

Proof. We have

$$\langle f|P_j\bar{P}_iP_l|f\rangle = \begin{cases} 1 & \text{if} \quad P_i = P_jP_l \\ 0 & \text{otherwise} \end{cases}$$

This follows since

$$\langle f|P_j\bar{P}_iP_l|f\rangle = \langle f|P_jP_lP_i^{-1}|f\rangle$$

($\bar{P}_i$ commutes with P_l and $\bar{P}_i = P_i^{-1}$), and $\langle f|$ and $|f\rangle$ are orthogonal, so $\langle f|P_jP_lP_i^{-1}|f\rangle$ has to equal unity, but can only do so if

$$P_jP_lP_i^{-1} = 1$$

which implies

$$P_jP_l = P_i$$

Thus, the expression $\langle f|P_j\bar{P}_iP_l|f\rangle$ reduces to a kind of Kronecker delta function:

$$\langle f|P_j\bar{P}_iP_l|f\rangle = \delta_{P_i,P_jP_l}$$

Using this and substituting for $\bar{P}_i$, we obtain

$$\begin{aligned} &\sum_i \langle f'|QP_i|f\rangle\langle f|P_j\bar{P}_iP_l|f\rangle \\ &= \sum_i \langle f'|QP_jP_l|f\rangle \\ &= \sum_i \langle f'|P_jQP_l|f\rangle \\ &= \langle f'|P_jQP_l|f\rangle \quad \blacksquare \end{aligned}$$

Thus we have shown that

$$\langle f'|P_jQP_l|f\rangle = \sum_i \langle f'|QP_i|f\rangle\langle f|P_j\bar{P}_iP_l|f\rangle$$

which, following Dirac, we can write as

$$Q \approx \sum_i c_i\bar{P}_i$$

where $c_i = \langle f'|QP_i|f\rangle$ and the sign $\approx$ means an equation in a restricted sense, the operators on the two sides being equal so long as they are used only with kets of the form $P|f\rangle$ and their conjugate imaginary bras.

If Q is taken to be some interaction energy V, we then have

$$\langle f'|P_jVP_l|f\rangle = \sum_i \langle f'|VP_i|f\rangle\langle f|P_j\bar{P}_iP_l|f\rangle$$

or

$$V \approx \sum_i c_i\bar{P}_i \tag{7}$$

Thus, the interaction energy is equal, in the restricted sense, to a linear function of the $\bar{P}$'s with coefficients c_i, given by $\langle f'|VP_i|f\rangle$.

We now assume that a two-body force only acts between the particles (such as the Coulomb force between electrons, to give an obvious example), so that the interaction energy V will consist of a sum of parts each referring to only two particles. This results in the vanishing of all matrix elements V *except* those for which $\bar{P}_i$ is either the identical permutation or an interchange of two "places," i.e., $\bar{P}_i = \bar{P}_1$ or $\bar{P}_2$.

Thus, for two-body forces V can be written as

$$V = V(a_1a_2) + V(a_2a_3) + V(a_1a_3)$$

where the terms in parentheses refer to place interchanges. Expression (7) then reduces to

$$V \approx V_1 + \sum_{r<s} V(a_ra_s)\bar{P}_{rs} \tag{8}$$

where $V(a_ra_s)$ is the matrix element referring to the interchange of "places" a_r and a_s occupied by the particles.

It should be noted that this assumption implies that we are only considering first-order perturbation terms and thus the eigenvalues of (8) will give only the first-order corrections to the transitions.

Granted this assumption, we can now write our expansion of V as

$$V = c_1\bar{P}_1 + c_2\bar{P}_2 + c_3\bar{P}_3 + c_4\bar{P}_4 + c_5\bar{P}_5 + c_6\bar{P}_6 \tag{9}$$

where $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_6$ are the place permutation operators defined in the usual way.

Now we need to show that certain terms in this expansion are zero. This means looking at the coefficients c_i, i.e., the first term in the product

$$\langle f'|VP_i|f\rangle\langle f|P_j\bar{P}_iP_i|f\rangle$$

since it is this that will in general be zero for certain $\bar{P}_i$. In the second term, for any value of $\bar{P}_i$ that might give us zero we can always find values of P_j

and P_l that give 1. In other words, there is no generality to be had from the second term because of the particular nature of P_j and P_l.

We therefore suppose that V is some suitable short-range interaction such that it cannot reach to Mars and thus cannot induce transitions between the eigenstates a^3 and $a^{3'}$ of the Mars-bound particle and eigenstates a^1, $a^{1'}$, a^2, $a^{2'}$ of the two located on Earth. Thus, all interaction terms containing either a^3 or $a^{3'}$ of the form $a_1^1 V(a_1 a_3) a_3^3$ or $a_3^{3'} V(a_1 a_3) a_1^1$ are zero. The only terms to survive are those for which both a^3 and $a^{3'}$ are outside the interaction term, which means that the only terms in equation (9) that are not zero are those containing interaction terms of the form $V(a_1 a_2)$ between $a^{1'}$, $a^{2'}$, a^1, a^2 only (see Appendix A). Thus, the only coefficients in our expansion that survive are c_1 and c_2 associated with $\bar{P}_1$ and $\bar{P}_2$, these being the only ones with interaction terms $V(a_1 a_2)$ of the form just described.

We therefore obtain

$$V \approx c_1 \bar{P}_1 + c_2 \bar{P}_2 \tag{10}$$

Thus, V commutes with $\bar{P}_2$ and we can write

$$[V, \bar{P}_2] = 0$$

Therefore for the particular case of the two-body collisions under discussion, there can be no transitions between states that possess opposite eigenvalues of the place permutation operators $\bar{P}_2$. Since the two subspaces of the same triangular representation are spanned by two sets of eigenvectors f'_{11}, f'_{21} and f''_{12}, f''_{22}, respectively, which possess opposite eigenvalues of $\bar{P}_2$, we conclude that transitions are forbidden between states corresponding to these subspaces. In other words, given these particular conditions, there can be no transitions between states corresponding to equivalent irreducible representations.

However, such transitions *may* occur in general, as we shall see directly.

The relevance of all this for Pauli's problem is as follows. If a third particle interacts with a two-particle subsystem, then transitions between bosonic and fermionic type states can occur in general, as Pauli suggested. However, if the third particle is located on Mars, then our discussion of the cluster principle above confirms that its mere existence cannot produce such transitions for two-particle systems on Earth.

We now go on to demonstrate how the general three-body collision (by which we mean collisions among three bodies, *not necessarily simultaneously*), *can* induce transitions between states corresponding to equivalent representations.

In this case, a_1^1, $a_1^{1'}$, a_3^3, $a_3^{3'}$ are taken to be the wave functions localized on Earth, with a_2^2, $a_2^{2'}$ on Mars. The cluster principle then dictates that terms of the form $a_1^1 a_1^2$ or $a_1^3 a_1^2$ are identically zero. Our results on p. 1153

are obviously still applicable and the interaction energy can be taken in the same form.

If V is again short range, then all interaction terms containing either a^2 or $a^{2'}$, i.e., of the form $a_1^1 V(a_1 a_2) a_2^2$ or $a_2^{2'} V(a_1 a_2) a_1^1$, are zero (see Appendix B). Thus, the only terms of the expansion to survive are those containing interaction terms of the form $V(a_1 a_3)$ between a^1, $a^{1'}$, a^3, $a^{3'}$ only. The only coefficients not equal to zero are therefore c_1 and c_3 associated with $\bar{P}_1$ and $\bar{P}_3$.

We therefore obtain

$$V \approx c_1 \bar{P}_1 + c_3 \bar{P}_3 \tag{11}$$

Since we chose our wave functions to be eigenfunctions of $\bar{P}_2$, we could conclude from our previous result (10) that transitions could not take place between states possessing opposite eigenvalues of $\bar{P}_2$. No such conclusion can be drawn from equation (11) because our wave functions are *not* eigenfunctions of $\bar{P}_3$ (French, 1985, p. 162 and pp. 178–179).

Thus, our two-particle cluster on Earth is in this case neither bosonic nor fermionic. Of course, we could always make it so, as before, through a unitary transformation that would make the wave functions eigenfunctions of $\bar{P}_3$, but obviously in that case they could not also be eigenfunctions of $\bar{P}_2$, since $\bar{P}_3$ and $\bar{P}_2$ do not commute.

We can conclude, therefore, that with three-body collisions transitions can occur between states corresponding to subspaces spanned by the vectors f'_{11}, f'_{21} and f''_{12}, f''_{22} through the collision of particle 1 with particle 3. In this case states corresponding to $\mathscr{E}_{t'}$ and $\mathscr{E}_{t''}$ can be connected. Thus, in a three-body collision involving particle 3 colliding with particle 1 followed by 2, the first collision can take the particles from a state supported by one subspace to a state supported by another, whereas the second can induce transitions between states supported by the same (the latter) subspace.

5. THE STATISTICS OF PARA-PARTICLES

These results clearly possess certain implications with regard to the weighting assignments in para-particle statistics. To determine these, we need to consider whether or not transitions are possible between our initial arrangement and some final one. The statistical weight given to the latter will then depend upon which of these transitions are allowed.

Although any given arrangement corresponds to two possible states represented by different subspaces of the same representation (e.g., $\mathscr{E}_{t'}$ and $\mathscr{E}_{t''}$), it is not necessary to consider transitions from *both* initial states, since the same results will obviously be obtained whichever pair of eigenfunctions

are taken. Also, we need consider only one vector out of either of the two pairs, since there are no transitions within a subspace.

Thus we choose f'_{11}, which we recall is given by

$$f'_{11} = \frac{1}{2\sqrt{3}} (2|a_1^1 a_2^2 a_3^3\rangle + 2|a_2^1 a_1^2 a_3^3\rangle - |a_1^1 a_3^2 a_2^3\rangle$$
$$- |a_3^1 a_2^2 a_1^3\rangle - |a_2^1 a_3^2 a_1^3\rangle - |a_3^1 a_1^2 a_2^3\rangle)$$

to represent our initial state, and the final one will be represented by the vector obtained by setting certain wave functions equal to one another, depending on the final state considered.

With one particle in each state, a^1, a^2, and a^3 are all different and the derived function is simply f'_{11}. In this case, with three-body collisions there exists the possibility of transitions taking place between states represented by f'_{11} and f'_{12}, giving a doubling of the allowed states and hence also the weight. Thus, in this case a weighing factor of 2 rather than 1 is assigned.

With all three particles in the same state we have $a^3 = a^2 = a^1$ and transitions to the final state do not occur, since the derived function equals zero. However, with two particles in one state, so that we have $a^3 = a^1$, for example, we obtain

$$f'_{11} = \frac{1}{2\sqrt{3}} (|a_1^1 a_2^2 a_3^1\rangle + |a_2^1 a_1^2 a_3^1\rangle - 2|a_1^1 a_3^2 a_2^1\rangle)$$

Similarly,

$$f''_{12} = \tfrac{1}{2}(-|a_1^1 a_2^2 a_3^1\rangle + 2|a_1^1 a_3^2 a_2^1\rangle - |a_2^1 a_1^2 a_3^1\rangle)$$

In this case there is no doubling of states, because f''_{12} is a multiple of f'_{11} and so there is only one set of transitions from the initial to the final state. Accordingly, a weight of only 1 is assigned for three-body collisions.

Continuing in this manner, we obtain a list of weights as shown in Table II. Included for comparison are the weights for particles obeying Gentile's (1961) para-gas statistics (see also Wergeland, 1944; Schubert, 1946; McCarthy, 1955). In a para-gas the average number of particles in a group of states is dependent on a parameter d giving the maximum number of particles that can occupy any given state. This is simply a form of generalized quantum statistics, with the Fermi-Dirac and Bose-Einstein cases resulting when $d = 1$ and $d = \infty$, respectively. In particular there is no doubling of weights due to transitions between subspaces supporting equivalent representations.

From Table II we get Table III, giving the distribution numbers and Tables IV and V, setting out quite explicitly the various probabilities for the different kinds of statistics. In the latter the statistical nature of both a

Table II. Weights

Final state	MB	FD	BE	Para-gas	Para-particle
$a^1 \neq a^2 \neq a^3$	6	1	1	1	2
$a^3 = a^1$	3	0	1	1	1
$a^2 = a^1$	3	0	1	1	1
$a^3 = a^2$	3	0	1	1	1
$a^1 = a^2$	3	0	1	1	1
$a^2 = a^3$	3	0	1	1	1
$a^1 = a^3$	3	0	1	1	1
$a^3 = a^2 = a^1$	1	0	1	0	0
$a^3 = a^1 = a^2$	1	0	1	0	0
$a^1 = a^2 = a^3$	1	0	1	0	0
Total weights	27	1	10	7	8

Table III. Distribution Numbers

Distribution number		MB	FD	BE	Para-gas	Para-particle
$n_a^1 =$	1	12	1	3	3	4
	2	6	0	2	2	2
	3	1	0	1	0	0
$n_a^2 =$	1	12	1	3	3	4
	2	6	0	2	2	2
	3	1	0	1	0	0
$n_a^3 =$	1	12	1	3	3	4
	2	6	0	2	2	2
	3	1	0	1	0	0

Table IV. Probabilities

Probability		MB	FD	BE	Para-gas	Para-particle
$n_a^1 =$	1	4/9	1	3/10	3/7	1/2
	2	2/9	0	2/10	2/7	1/4
	3	1/27	0	1/10	0	0
$n_a^2 =$	1	4/9	1	3/10	3/7	1/2
	2	2/9	0	2/10	2/7	1/4
	3	1/27	0	1/10	0	0
$n_a^3 =$	1	4/9	1	3/10	3/7	1/2
	2	2/9	0	2/10	2/7	1/4
	3	1/27	0	1/10	0	0

Table V. Probabilities

Probability	MB	FD	BE	Para-gas	Para-particle
One in each state	2/9	1	1/10	1/7	1/4
Two in any state	2/3	0	6/10	6/7	3/4
Three in any state	1/9	0	3/10	0	0
At least one in any state	19/27	1	6/10	5/7	3/4

para-gas and of para-particles is clearly revealed and we can see that the notion that these are merely "intermediate" statistics of some kind is too simplistic. Thus, for both cases, the probability of finding one particle in any state lies somewhere between that for Bose–Einstein and Fermi–Dirac statistics, but, on the other hand, the tendency for two particles to cluster together in any state is greater than for bosons, (this is not true of course for the three-particle case).

However, the more interesting comparison is between para-gas and para-particle statistics themselves. First of all, the total number of allowed states is smaller for the former than for the latter. Second, although the probability of finding two particles in any state is greater for the para-gas case, the probability of finding one particle in any state and that of finding one in each state is greater for para-particles. Furthermore, there is a greater probability of finding at least one particle in any of the three states in para-particle statistics as compared to those of the para-gas. We can conclude from all this that the statistical behavior of a para-gas and that of an assembly of para-particles differ in that the former exhibit a greater tendency to cluster together than the latter, or, conversely, that the latter exhibit a statistical repulsion as compared to the former.

6. CONCLUSION

In their seminal paper, Hartle and Taylor (1969, p. 2051) remarked that, "although there is no theoretical reason to exclude para-particles, their properties are sufficiently disagreeable for one to hope sincerely that there will continue to be no evidence in their favour."

The simple discussion above may serve to suggest that this appraisal of the situation was perhaps a little too pessimistic and that the investigation of some of the properties of these entities is not quite as unapproachable as is often supposed. Beyond this I hope to have shown that the general formalism possesses a certain intrinsic elegance and may provide further insight into the statistical behavior of N-particle systems.

APPENDIX A

As an example, consider the coefficient c_3 associated with $\bar{P}_3$ in expansion (9):

$$\langle f'|VP_3|f\rangle = \int\int\int f'^* VP_3 f\, da_1\, da_2\, da_3$$

Expanding V, $|f\rangle$, and $|f'\rangle$, we have

$$\begin{aligned}
&\langle f'|VP_3|f\rangle \\
&= \int\int\int a_1^{1'*} a_2^{2'*} a_3^{3'*} [V(a_1a_2) + V(a_2a_3) + V(a_1a_3)] \\
&\quad \times P_3 a_1^1 a_2^2 a_3^3\, da_1\, da_2\, da_3 \\
&= \int\int\int a_1^{1'*} a_2^{2'*} a_3^{3'*} [V(a_1a_2) + V(a_2a_3) + V(a_1a_3)] \\
&\quad \times a_3^1 a_2^2 a_1^3\, da_1\, da_2\, da_3 \\
&= \int\int\int a_3^{3'*} a_3^1 [a_1^{1'*} a_2^{2'*} V(a_1a_2) a_2^2 a_1^3]\, da_1\, da_2\, da_3 \\
&\quad + \int\int\int a_1^{1'*} a_1^3 [a_2^{2'*} a_3^{3'*} V(a_2a_3) a_3^1 a_2^2]\, da_1\, da_2\, da_3 \\
&\quad + \int\int\int a_2^{2'*} a_2^2 [a_1^{1'*} a_3^{3'*} V(a_1a_3) a_3^1 a_1^3]\, da_1\, da_2\, da_3 \\
&= \int\int\int a_3^{3'*} a_3^1 [a_1^{1'*} a_1^3 V(a_1a_2) a_2^{2'*} a_2^2]\, da_1\, da_2\, da_3 \\
&\quad + \int\int\int a_1^{1'*} a_1^3 [a_2^{2'*} a_2^2 V(a_2a_3) a_3^{3'*} a_3^1]\, da_1\, da_2\, da_3 \\
&\quad + \int\int\int a_2^{2'*} a_2^2 [a_1^{1'*} a_1^3 V(a_1a_3) a_3^{3'*} a_3^1]\, da_1\, da_2\, da_3
\end{aligned}$$

Now consider each of these terms in turn. In the first, one has

$$a_1^{1'*} a_1^3 V(a_1a_2) a_2^{2'*} a_2^2$$

but V is short-range and cannot connect any eigenstates with a^3 or $a^{3'}$. Thus, $V(a_1a_2)$ cannot reach between a_1^3 and a_2^2 and so this term is zero. Similarly,

$$a_2^{2'*} a_2^2 V(a_2a_3) a_3^{3'*} a_3^1$$

and

$$a_1^{1'*} a_1^3 V(a_1a_3) a_3^{3'*} a_3^1$$

are also zero because of the nature of V, and thus we obtain

$$\langle f'|VP_3|f\rangle = 0$$

or $c_3 = 0$.

In the same way it can be shown that c_4, c_5, and c_6 are also identically zero.

However, c_1 and c_2 contain terms for which both a^3 and $a^{3'}$ fall outside the interaction term, thus giving finite nonzero values to these terms. Consider, for example, c_2:

$$\langle f'|VP_2|f\rangle = \int\int\int f'^* VP_2 f\, da_1\, da_2\, da_3$$

Expanding gives

$$\begin{aligned}
&\langle f'|VP_2|f\rangle \\
&= \int\int\int a_1^{1'*} a_2^{2'*} a_3^{3'*}[V(a_1a_2)+V(a_2a_3)+V(a_1a_3)] \\
&\quad \times P_2 a_1^1 a_2^2 a_3^3\, da_1\, da_2\, da_3 \\
&= \int\int\int a_1^{1'*} a_2^{2'*} a_3^{3'*}[V(a_1a_2)+V(a_2a_3)+V(a_1a_3)] \\
&\quad \times a_2^1 a_1^2 a_3^3\, da_1\, da_2\, da_3 \\
&= \int\int\int a_3^{3'*} a_3^3[a_1^{1'*} a_2^{2'*} V(a_1a_2) a_2^1 a_1^2]\, da_1\, da_2\, da_3 \\
&\quad + \int\int\int a_1^{1'*} a_1^2[a_2^{2'*} a_3^{3'*} V(a_2a_3) a_2^1 a_3^3]\, da_1\, da_2\, da_3 \\
&\quad + \int\int\int a_2^{2'*} a_2^1[a_1^{1'*} a_3^{3'*} V(a_1a_3) a_1^2 a_3^3]\, da_1\, da_2\, da_3 \\
&= \int\int\int a_3^{3'*} a_3^3[a_1^{1'*} a_1^2 V(a_1a_2) a_2^1 a_2^{2'*}]\, da_1\, da_2\, da_3 \\
&\quad + \int\int\int a_1^{1'*} a_1^2[a_2^{2'*} a_2^1 V(a_2a_3) a_3^{3'*} a_3^3]\, da_1\, da_2\, da_3 \\
&\quad + \int\int\int a_2^{2'*} a_2^2[a_1^{1'*} a_1^2 V(a_1a_3) a_3^{3'*} a_3^3]\, da_1\, da_2\, da_3
\end{aligned}$$

It can immediately be seen that the interaction term

$$a_1^{1'*} a_1^2 V(a_1a_2) a_2^1 a_2^{2'*}$$

is nonzero, as the short-range characteristic of V certainly does allow $V(a_1a_2)$ to induce transitions between $a_1^{1'*}$ and $a_2^{2'*}$ or a_1^2 and a_2^1. However, the expression as a whole can only be finite if $a^{3'} = a^3$. This follows of course from the orthogonality of these functions. Thus, $a^{3'}$ and a^3 must not only overlap in order for c_2 to be nonzero, they must in fact be equal.

The second term above contains

$$a_2^{2'*} a_2^1 V(a_2a_3) a_3^{3'*} a_3^3$$

which is zero, since $V(a_2a_3)$ cannot connect $a_2^{2'*}$ with $a_2^{3'*}$ or a_2^1 with a_3^3. Similarly, the third term above can be shown to be zero, and we thus obtain

$$c_2 = \text{finite} \qquad \text{if} \quad a^{3'} = a^3$$

It can be shown in the same way that the only term in c_1 that is finite is that containing $V(a_1a_2)$ and then only if $a^{3'} = a^3$.

APPENDIX B

Consider now the coefficient c_2 associated with $\bar{P}_2$:

$$\langle f' | VP_2 | f \rangle$$

$$= \iiint a_1^{1'*} a_2^{2'*} a_3^{3'*} [V(a_1a_2) + V(a_2a_3) + V(a_1a_3)]$$

$$\times P_2 a_1^1 a_2^2 a_3^3 \, da_1 \, da_2 \, da_3$$

Expanding as before, we obtain

$$\langle f' | VP_2 | f \rangle$$

$$= \iiint a_1^{1'*} a_2^{2'*} a_3^{3'*} [V(a_1a_2) + V(a_2a_3) + V(a_1a_3)]$$

$$\times a_2^1 a_1^2 a_3^3 \, da_1 \, da_2 \, da_3$$

$$= \iiint a_3^{3'*} a_3^3 [a_1^{1'*} a_1^2 V(a_1a_2) a_2^1 a_2^{2'*}] \, da_1 \, da_2 \, da_3$$

$$+ \iiint a_1^{1'*} a_1^2 [a_2^{2'*} a_1^{1'*} V(a_2a_3) a_3^{3'*} a_3^3] \, da_1 \, da_2 \, da_3$$

$$+ \iiint a_2^{2'*} a_2^2 [a_1^{1'*} a_1^2 V(a_1a_3) a_3^{3'*} a_3^3] \, da_1 \, da_2 \, da_3$$

Again the short-range nature of V implies that the interaction terms in these expressions are all zero.

However, c_1 and c_3 contain terms for which both a^2 and $a^{2'}$ fall outside the interaction term, giving finite nonzero values. Expanding c_3 gives

$$\begin{aligned}
&\langle f'|VP_3|f\rangle \\
&\quad = \int\int\int a_1^{1'*} a_2^{2'*} a_3^{3'*} [V(a_1a_2) + V(a_2a_3) + V(a_1a_3)] \\
&\qquad \times a_3^1 a_2^2 a_1^3 \, da_1 \, da_2 \, da_3 \\
&\quad = \int\int\int a_3^{3'*} a_3^1 [a_1^{1'*} a_2^2 V(a_1a_2) a_1^3 a_2^{2'*}] \, da_1 \, da_2 \, da_3 \\
&\qquad + \int\int\int a_1^{1'*} a_1^3 [a_2^{2'*} a_2^2 V(a_2a_3) a_3^{3'*} a_3^1] \, da_1 \, da_2 \, da_3 \\
&\qquad + \int\int\int a_2^{2'*} a_2^2 [a_1^{1'*} a_1^3 V(a_1a_3) a_3^1 a_3^{3'*}] \, da_1 \, da_2 \, da_3
\end{aligned}$$

and now

$$a_1^{1'*} a_1^3 V(a_1a_3) a_3^1 a_3^{3'*}$$

is nonzero. Repeating this procedure for the other terms, we obtain

$$V \approx c_1 \bar{P}_1 + c_3 \bar{P}_3$$

as required.

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